Sidebar: a Singular Vlasov Equation

Take a fermionic condensate $f(t,x,v) = \mathbb{1}_{\{|v-u| \leq \rho^{1/d}\}}$

where $\rho = \frac{d}{|\mathbb{S}^{d-1}|} \int f \, dv$ and $\rho u = \frac{d}{|\mathbb{S}^{d-1}|} \int f \, v \, dv$.

are smooth solutions of

$$\boxed{\begin{cases} \partial_t \rho + \mathrm{div}_x (\rho u) = 0 \\ \partial_t \rho u + \mathrm{div}_x (\rho u \otimes u) + \frac{1}{d+2} \nabla_x \left(\rho^{1+\frac{2}{d}}\right) = 0 \end{cases}}$$

Then, at least formally, $f(t,x,v)$ solves the Vlasov equation

$$\boxed{\left(\partial_t + v \cdot \nabla_x\right) f(t,x,v) + (v-u) \cdot \left(\frac{\nabla_x u + \nabla_x^t u}{2} - \frac{\mathrm{div}_x u}{d} \mathrm{Id}\right) \nabla_v f(t,x,v) = 0}$$

viscous stress tensor

- It's a very singular Vlasov equation. In the spirit of the Vlasov-Dirac-Benney equation (force $= \nabla_x \rho$).
- Is the fermionic condensate solution stable?
- Physical significance is unclear. It's an equation!

The Entropy Dissipation near Absolute Zero

The entropy dissipation bound: $(\tau > 0$ and $k > 2\tau)$

$$\frac{1}{\varepsilon^{2+k-\tau}} \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_{FD}(f_\varepsilon)(s)\, dx\, ds \leq C^{in} < \infty$$

$$\mathcal{D}_{FD}(f) = -\delta \int Q_{FD}(f)(v) \log\left(\frac{\delta f}{1-\delta f}\right) dv$$

$$= \frac{\delta}{4} \int \Big(f' f'_*(1-\delta f)(1-\delta f_*) - f f_*(1-\delta f')(1-\delta f'_*) \Big) \log\left(\frac{f' f'_*(1-\delta f)(1-\delta f_*)}{f f_*(1-\delta f')(1-\delta f'_*)} \right) b\, dv\, dv_*\, d\sigma \geq 0$$

Employing the elementary inequality

$$4(\sqrt{y} - \sqrt{z})^2 \leq (y-z) \log\left(\frac{y}{z}\right)$$

we find

$$\frac{\delta}{\varepsilon^{2+k-\tau}} \int \left(\sqrt{f' f'_*(1-\delta f)(1-\delta f_*)} - \sqrt{f f_*(1-\delta f')(1-\delta f'_*)} \right)^2 b\, dv\, dv_*\, d\sigma\, dx\, ds \leq C^{in}$$

$\Rightarrow$ Introduce renormalized fluctuations:

$$\sqrt{\delta f_\varepsilon} = \sqrt{\delta M_\varepsilon}\left(1 + \frac{\delta \varepsilon}{2} \varphi_\varepsilon\right) \quad \text{and} \quad \sqrt{1-\delta f_\varepsilon} = \sqrt{1-\delta M_\varepsilon}\left(1 - \frac{\delta \varepsilon}{2} \varphi_\varepsilon\right)$$

$$\Rightarrow \quad g_\varepsilon = \delta M_\varepsilon(1-\delta M_\varepsilon)(\phi_\varepsilon + \psi_\varepsilon) + \frac{\delta\varepsilon}{4}\delta M_\varepsilon(1-\delta M_\varepsilon)(\phi_\varepsilon^2 - \psi_\varepsilon^2)$$

$$= O\left(\varepsilon^{\frac{\tau-\gamma}{2}}\right)_{L^\infty(dt;L^2(dx;L^1(dv)))} + O\left(\varepsilon^{1-\gamma}\right)_{L^\infty(dt;L^1(dxdv))}$$

(by the relative entropy bound) (recall that $0<\tau=\gamma<1$)

<u>Notation:</u> $m_\varepsilon = \delta M_\varepsilon \delta M_{\varepsilon*}(1-\delta M_\varepsilon')(1-\delta M_{\varepsilon*}') = \delta M_\varepsilon' \delta M_{\varepsilon*}'(1-\delta M_\varepsilon)(1-\delta M_{\varepsilon*})$

$$\tilde{\Pi}_\varepsilon = \phi_\varepsilon + \psi_\varepsilon \quad \Rightarrow \quad g_\varepsilon = \delta M_\varepsilon(1-\delta M_\varepsilon)\tilde{\Pi}_\varepsilon + \text{small}$$

<u>Lemma:</u> The relative entropy bound and entropy dissipation bound imply that

$$\tilde{\Pi}_\varepsilon + \tilde{\Pi}_{\varepsilon*} - \tilde{\Pi}_\varepsilon' - \tilde{\Pi}_{\varepsilon*}' = O\left(\varepsilon^{1+2\tau-\gamma}\right)_{L^\infty(dt;L^1(b m_\varepsilon dxdvdv_* d\sigma))}$$

$$+ O\left(\varepsilon^{\frac{k-\gamma+3\tau}{2}}\right)_{L^2(dtdx;L^1(b m_\varepsilon dvdv_* d\sigma))}$$

if $d\geq 3$, and

$$\tilde{\Pi}_\varepsilon + \tilde{\Pi}_{\varepsilon*} - \tilde{\Pi}_\varepsilon' - \tilde{\Pi}_{\varepsilon*}' = O\left(\varepsilon^{1+2\tau-\gamma}|\log\varepsilon|\right)_{L^\infty(dt;L^1(b m_\varepsilon dxdvdv_* d\sigma))}$$

$$+ O\left(\varepsilon^{\frac{k-\gamma+3\tau}{2}}|\log\varepsilon|^{\frac{1}{2}}\right)_{L^2(dtdx;L^1(b m_\varepsilon dvdv_* d\sigma))}$$

if $d=2$.

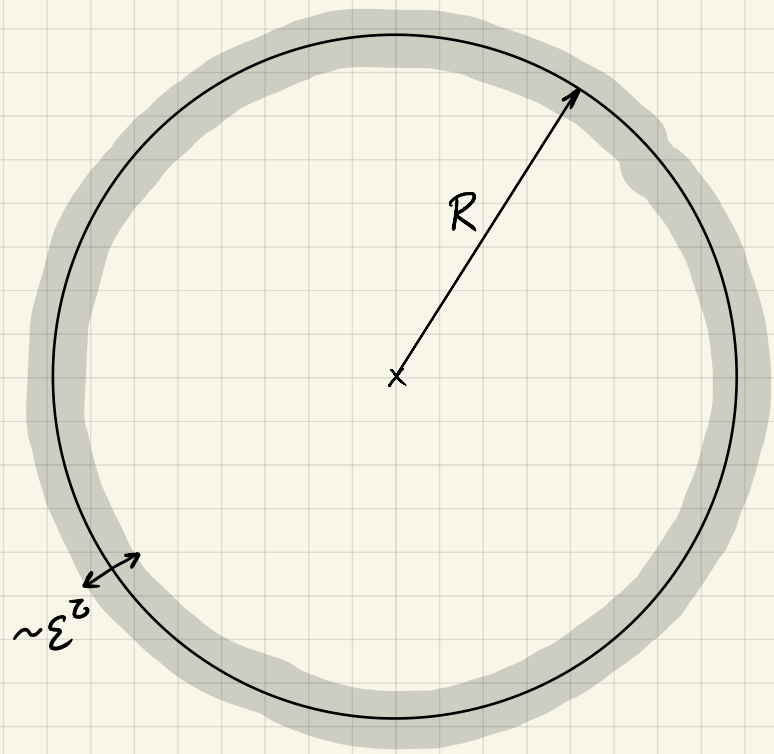
In particular, if $0<\gamma=\tau<1$ and $k>2\tau$, then

$$\frac{1}{\varepsilon^{2\tau}}\left(\tilde{\Pi}_\varepsilon + \tilde{\Pi}_{\varepsilon*} - \tilde{\Pi}_\varepsilon' - \tilde{\Pi}_{\varepsilon*}'\right) = o(1)_{L^1_{loc}(dtdx;L^1(b m_\varepsilon dvdv_* d\sigma))}$$

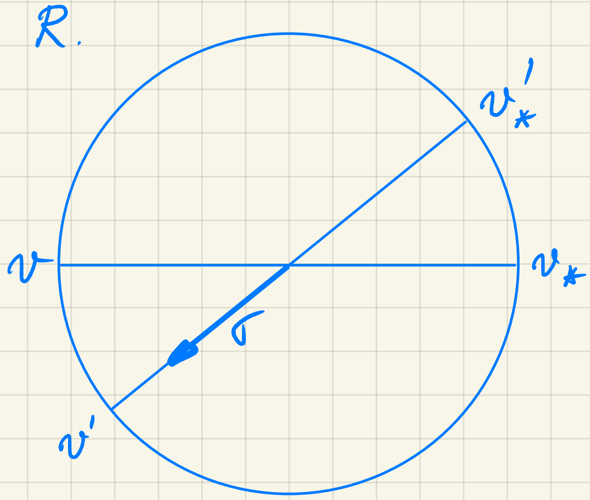
Relaxation toward Thermodynamic Equilibrium

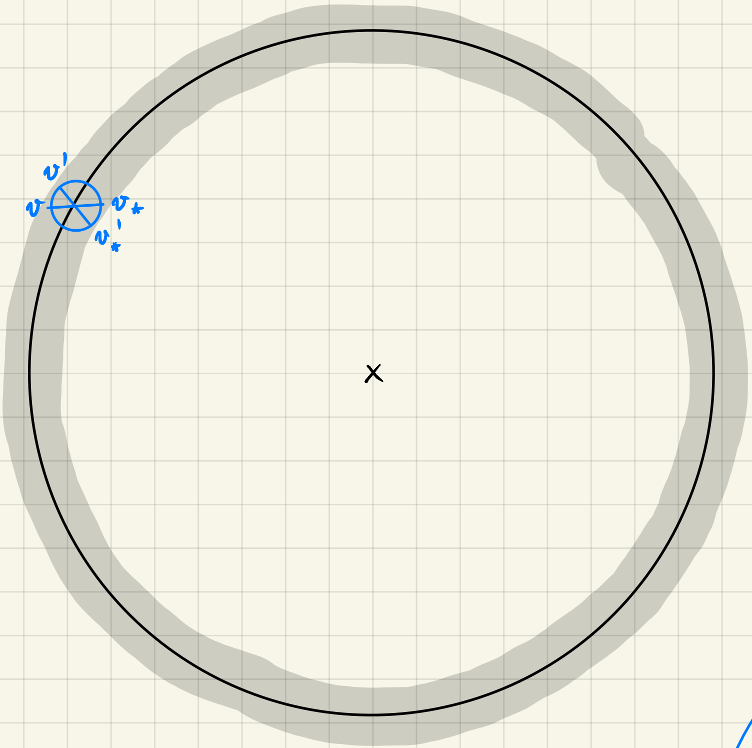
$$\boxed{\begin{aligned} m_\varepsilon &= \delta M_\varepsilon \, \delta M_{\varepsilon *} (1-\delta M_\varepsilon')(1-\delta M_{\varepsilon *}') = \delta M_\varepsilon' \, \delta M_{\varepsilon *}' (1-\delta M_\varepsilon)(1-\delta M_{\varepsilon *}) \\ &= \sqrt{\delta M_\varepsilon (1-\delta M_\varepsilon)\, \delta M_{\varepsilon *} (1-\delta M_{\varepsilon *})\, \delta M_\varepsilon' (1-\delta M_\varepsilon')\, \delta M_{\varepsilon *}' (1-\delta M_{\varepsilon *}')} \end{aligned}}$$

What does m_ε do? It concentrates all velocities on a sphere.



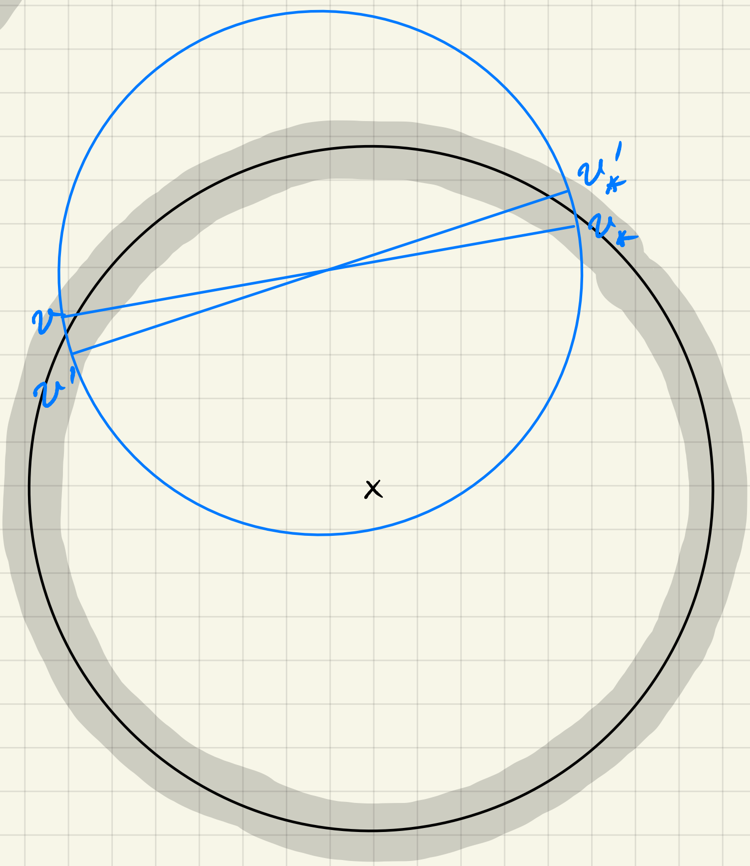
Game: Try to confine pre-collisional and post-collisional velocities (v, v_*, v', v_*') to a layer of thickness ε^τ of the sphere of radius R.

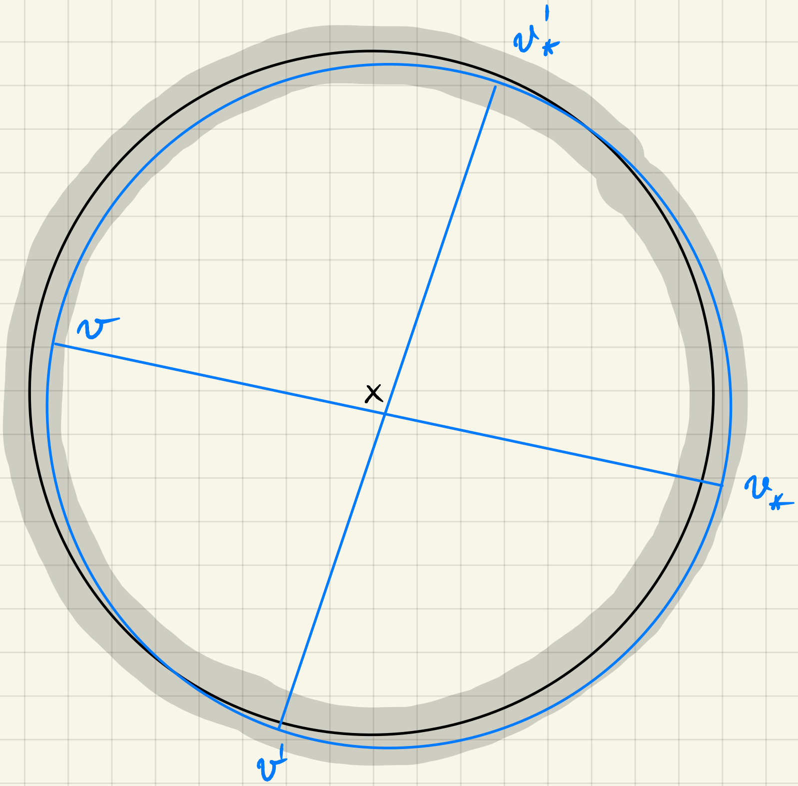




$\Rightarrow$ as $\varepsilon \to 0$, $v \sim v_* \sim v' \sim v'_*$

as $\varepsilon \to 0$, $v \sim v'$ and $v_* \sim v'_*$ (or $v \sim v'_*$ and $v_* \sim v'$) $\Leftarrow$





$\Rightarrow$ as $\varepsilon \to 0$,

$v_* \sim -v$

$v' \sim -v'_*$

As $\varepsilon \to 0$,

$$g_\varepsilon \cong \delta M_\varepsilon (1-\delta M_\varepsilon)\, \tilde{\Pi}_\varepsilon \xrightarrow{\ *\ } g(t,x,\omega)\, dt \otimes dx \otimes \delta_{\partial B(0,R)}(\omega)$$

$\Rightarrow$ Formally, we expect

$$\boxed{g(\omega) + g(-\omega) - g(\omega_*) - g(-\omega_*) = 0 \ , \text{ for all } \omega, \omega_* \in \partial B(0,R).}$$

$\Rightarrow$ Thermodynamic equilibrium: even part $g_{even} = \dfrac{g(\omega)+g(-\omega)}{2}$

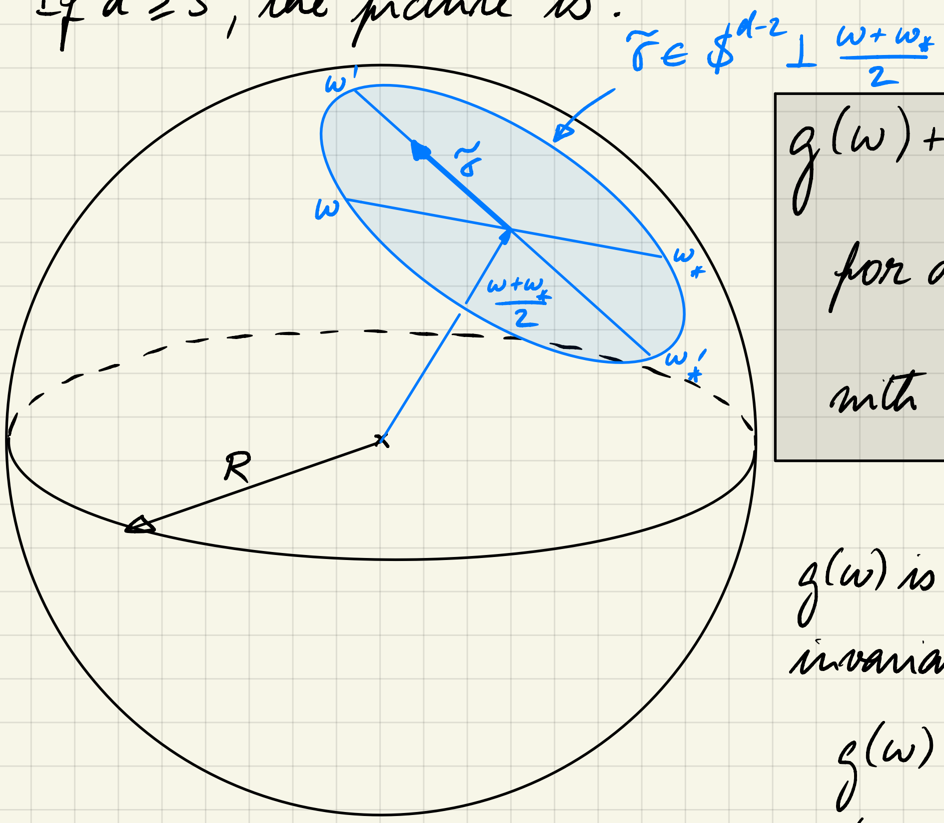
is constant in $\omega \in \partial B(0,R)$, but there is no constraint

on the odd part $g_{odd} = \dfrac{g(\omega)-g(-\omega)}{2}$.

WRONG! if $d \geqslant 3$.

This is the two-dimensional picture, only.

If $d \geq 3$, the picture is:



$$\boxed{\begin{array}{l} g(\omega) + g(\omega_*) - g(\omega') - g(\omega'_*) = 0 \\ \text{for all } \omega, \omega_*, \omega', \omega'_* \in \partial B(0,R) \\ \text{with } \omega + \omega_* = \omega' + \omega'_*. \end{array}}$$

$\Downarrow$

$g(\omega)$ is a quantized collision invariant on the sphere:

$$g(\omega) = \rho + u \cdot \omega$$

(requires a proof)

Formal convergence of the linearized operator ($d \geq 3$):

$$I_\varepsilon = \varepsilon^{-2\tau} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} \left(\Pi_\varepsilon + \Pi_{\varepsilon *} - \Pi_\varepsilon' - \Pi_{\varepsilon *}' \right) \varphi\left(\frac{v^2 - R^2}{\varepsilon^\tau}, v \right) b \, m_\varepsilon^\alpha \, dv \, dv_* \, d\sigma \quad (\alpha > 1)$$

$$= \varepsilon^{-2\tau} \int \Pi_\varepsilon \left(\varphi + \varphi_* - \varphi' - \varphi_*' \right) b \, m_\varepsilon^\alpha \, dv \, dv_* \, d\sigma$$

Introduce suitable variables:

$$u = \frac{v^2 - R^2}{\varepsilon^\tau}, \quad u_* = \frac{v_*^2 - R^2}{\varepsilon^\tau}, \quad \nu = \frac{v}{|v|}, \quad \nu_* = \frac{v_*}{|v_*|}$$

$$\cos\eta = \frac{v + v_*}{|v + v_*|} \cdot \sigma, \qquad s = \frac{|v + v_*| \, |v - v_*|}{2 \varepsilon^\tau} \cos\eta$$

$$\sigma = \frac{v + v_*}{|v + v_*|} \cos\eta + \tilde{\sigma} \sin\eta, \text{ with } \tilde{\sigma} \in \mathbb{S}^{d-2} \perp \frac{v + v_*}{2}.$$

$$\Rightarrow I_\varepsilon = \int \frac{\tilde{\Pi}_\varepsilon \left(\int_{\mathbb{S}^{d-2} \perp \frac{v+v_*}{2}} \left(\varphi + \varphi_* - \varphi' - \varphi_*' \right) b \, d\tilde{\sigma} \right) \left(R^2 + \varepsilon^\tau u_* \right)_+^{\frac{d-2}{2}}}{8^\alpha \cosh^\alpha\left(\frac{u}{2}\right) \cosh^\alpha\left(\frac{u_*}{2}\right) \left(\cosh\left(\frac{u + u_*}{2}\right) + \cosh(s) \right)^\alpha}$$

$$\times \frac{\left(\left(|v + v_*| \, |v - v_*| \right)^2 - \left(2 \varepsilon^\tau s \right)^2 \right)^{\frac{d-3}{2}}}{\left(|v + v_*| \, |v - v_*| \right)^{d-2}} \, ds \, d\nu_* \, du_* \, d\nu \, du$$

where $$\boxed{\tilde{\Pi}_\varepsilon (u, \nu) = \frac{\varepsilon^\tau}{2} \Pi_\varepsilon(\nu) \left(R^2 + \varepsilon^\tau u \right)_+^{\frac{d-2}{2}}}$$

Assuming that $\widetilde{\pi}_\varepsilon \xrightharpoonup{*} \widetilde{\pi}$ (in a suitable sense),

one can show that $\boxed{g(Rv) = \int_{\mathbb{R}} \frac{\widetilde{\pi}(u,v)}{4R^{d-1}\cosh^2\left(\frac{u}{2}\right)}\, du}$

Furthermore, $I_\varepsilon \to I_0$ where (needs a serious proof)

$$I_0 = \int_{\mathbb{R}\times\mathbb{S}^{d-1}\times\mathbb{R}\times\mathbb{S}^{d-1}\times\mathbb{R}} \frac{\widetilde{\pi}(u,v)\int_{\mathbb{S}^{d-2}\perp\frac{v+v_*}{2}}\Big(\varphi(u,Rv)+\varphi(u_*,Rv_*)-\varphi(u',Rv')-\varphi(u_*',Rv_*')\Big)b(R(v-v_*),\tilde{\sigma})\,d\tilde{\sigma}}{8^\alpha\cosh^\alpha\left(\frac{u}{2}\right)\cosh^\alpha\left(\frac{u_*}{2}\right)\left(\cosh\left(\frac{u+u_*}{2}\right)+\cosh(s)\right)^\alpha |v+v_*|\,|v-v_*|}$$

$$\times R^{d-4}\, du_*\, dv_*\, du\, dv_*\, ds$$

with $u' = \frac{u+u_*}{2} + s$, $u_*' = \frac{u+u_*}{2} - s$,

$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\tilde{\sigma}$, $v_*' = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\tilde{\sigma}$.

Important: $\int_{\mathbb{S}^{d-1}} \frac{1}{|v+v_*|\,|v-v_*|}\, dv_* < \infty$ if and only if $d \geq 3$.

One can then show that $\Rightarrow \widetilde{\pi} + \widetilde{\pi}_* - \widetilde{\pi}' - \widetilde{\pi}_*' = 0$

$\Rightarrow \widetilde{\pi}(u,v) = A + B\cdot v + Cu$

$\Rightarrow \boxed{g(w) = \rho + u\cdot w}$ (proof is needed; does not follow from collision invariants in $\mathbb{R}^d$)

The Acoustic Limit near Fermionic Condensates

Suppose now that f_ε is a solution (à la Dolbeault) of

$$(\partial_t + v\cdot\nabla_x) f_\varepsilon = \frac{1}{\varepsilon^k} Q_{FD}(f_\varepsilon) \qquad (k > 2\tau)$$

with $\sup_{\varepsilon>0} \frac{1}{\varepsilon^{2-\tau}} H_{FD}(f_\varepsilon^{in} \mid M_\varepsilon) < \infty$.

We have shown

$$g_\varepsilon(t,x,v) \overset{*}{\rightharpoonup} g(t,x,\omega)\, dt \otimes dx \otimes \delta_{\partial B(0,R)}(\omega)$$

with $g(t,x,\omega) = \rho(t,x) + u(t,x)\cdot\omega \qquad (d \geq 3)$

and $\rho, u \in L^\infty(dt; L^2(dx))$.

Letting $\varepsilon \to 0$ in the local conservation laws, we see that

$$\partial_t \int_{\mathbb{R}^d} f_\varepsilon \, dv + \mathrm{div}_x \int_{\mathbb{R}^d} f_\varepsilon v \, dv = 0 \longrightarrow \partial_t \int_{\partial B(0,R)} g \, d\omega + \mathrm{div}_x \int_{\partial B(0,R)} g\, \omega \, d\omega = 0$$

$$\partial_t \int_{\mathbb{R}^d} f_\varepsilon v \, dv + \mathrm{div}_x \int_{\mathbb{R}^d} f_\varepsilon v \otimes v \, dv = 0 \longrightarrow \partial_t \int_{\partial B(0,R)} g\, \omega \, d\omega + \mathrm{div}_x \int_{\partial B(0,R)} g\, \omega \otimes \omega \, d\omega = 0$$

$\Rightarrow$ Up to multiplicative constants, we obtain the system:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x u = 0 \\ \partial_t u + \nabla_x \rho = 0 \end{cases}$$

If $d = 2$, we have the same result by relying on $g_{even} = \dfrac{g(\omega) + g(-\omega)}{2} = \rho(t,x)$. $\blacksquare$

Research Perspectives

- Hydrodynamic limits for more general collision kernels.
- Spectral gap estimates for the linearized operators.
- Isentropic Euler limit in 1-dimension.
- More singular low-temperature hydrodynamic limit $\tau = \gamma = 1$.
- Low-temperature hydrodynamic limit for other equation:
    - Boltzmann equation
    - Boltzmann-Bose-Einstein equation
    - BGK-model (for a fermionic-condensate density, for example)
- Incompressible Navier-Stokes limit near absolute zero.
- Incompressible Euler limit near absolute zero.
- Stability of Fermionic condensates in the BFD equation.