Macroscopic conservation laws for a Fermionic condensate:

Note that: $\rho = \int F(v)\,dv = \dfrac{|\mathbb{S}^{d-1}|\,R^d}{d\,\delta}$

$$\rho u = \int F(v)\, v\, dv = \frac{|\mathbb{S}^{d-1}|\,R^d}{d\,\delta} u$$

$$\rho\theta = \int F(v) \frac{|v-u|^2}{d} dv = \frac{|\mathbb{S}^{d-1}|\,R^{d+2}}{d(d+2)\delta} \Rightarrow \rho\theta \sim \rho^{1+2/d}$$

$\Rightarrow$ For a Fermionic condensate, the compressible Euler equations become the equations of isentropic gas dynamics:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla_x \cdot \left(\rho u \otimes u + C \rho^{1+2/d} \mathbb{I}d \right) = 0 , \text{ for some } C>0. \end{cases}$$

The corresponding acoustic waves system is:

$$\begin{cases} \partial_t \tilde{\rho} + \nabla_x \cdot \tilde{u} = 0 \\ \partial_t \tilde{u} + \nabla_x (C\tilde{\rho}) = 0 , \text{ for some } C>0. \end{cases}$$

Problem: There is no known mechanism ensuring the stability of Fermionic condensates (as Knudsen number $\varepsilon \to 0$).

Alternative: Study the acoustic limit of the Boltzmann–Fermi–Dirac equation near a global Fermionic condensate $F(v) = \delta^{-1} \mathbb{1}_{B(0,R)}(v)$.

Formal attempt of derivation of acoustic waves near a Fermionic condensate

$$f_\varepsilon = F + \varepsilon g_\varepsilon \quad , \quad F(v) = \delta^{-1} \mathbb{1}_{B(0,R)}(v)$$

$$(\partial_t + v \cdot \nabla_x) f_\varepsilon = \frac{1}{\varepsilon^k} Q_{FD}(f_\varepsilon) \quad , \quad 0 \leq f_\varepsilon \leq \delta^{-1}$$

Can we expect $g_\varepsilon = O(1)$?

How can we control the size of g_ε?

Use $H_{FD}(f_\varepsilon | F) + \frac{1}{\varepsilon^k} \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_{FD}(f_\varepsilon)(s)\, dx\, ds \leq H_{FD}(f_\varepsilon^{in} | F)$?

Problem: The relative entropy

$$H_{FD}(f|F) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\delta f \log\left(\frac{\delta f}{\delta F}\right) + (1-\delta f) \log\left(\frac{1-\delta f}{1-\delta F}\right)\right) dx\, dv$$

is not defined relatively to a Fermionic condensate!

Alternative: Study the acoustic limit of the Boltzmann-Fermi-Dirac equation near an equilibrium state

$$M_\varepsilon(v) = \frac{\delta^{-1}}{1 + \exp\left(\frac{v^2 - R^2}{\varepsilon^2}\right)} \quad , \quad R>0, \ \delta>0$$

with a temperature reaching absolute zero, as $\varepsilon \to 0$.

<u>Theorem:</u> Consider a cross-section $b(z,\sigma)$ such that

$$b(z,\sigma) \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1}) \cap C(B(0,2R)\times \mathbb{S}^{d-1})$$
$$b > 0 \quad \text{on } B(0,2R)\times\mathbb{S}^{d-1},$$

and a family of density distributions $0 \le f_\varepsilon(t,x,v) \le \delta^{-1}$, $\varepsilon>0$, such that

$$\boxed{\frac{1}{\varepsilon^{2-\tau}} H_{FD}(f_\varepsilon \mid M_\varepsilon) + \frac{1}{\varepsilon^{2+k-\tau}} \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_{FD}(f_\varepsilon)(s)\, dx\, ds \le C^{in} < \infty \quad \text{where } 0<\tau<1 \text{ and } k>2\tau.}$$

Then, as $\varepsilon \to 0$, up to extraction of a subsequence, the family of fluctuations g_ε given by

$$\boxed{f_\varepsilon = M_\varepsilon + \varepsilon g_\varepsilon}$$

converges in the weak* topology of $\mathcal{M}_{loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d)$ toward a limit point

$$\boxed{\mu(t,x,v) = g\left(t,x,R\frac{v}{|v|}\right) dt \otimes dx \otimes \delta_{\partial B(0,R)}}$$

with $g(t,x,\omega) \in L^\infty(dt; L^2(\mathbb{R}^d \times \partial B(0,R)))$.

Moreover, if $d \geq 3$, it holds that

$$\boxed{g(t,x,\omega) = \rho(t,x) + u(t,x)\cdot\omega}$$

with $\rho, u \in L^\infty(dt; L^2(\mathbb{R}^d))$.

If $d=2$, one has that

$$\boxed{\frac{g(t,x,\omega)+g(t,x,-\omega)}{2} = \rho(t,x)}$$

with no constraint on the odd part $\dfrac{g(t,x,\omega)-g(t,x,-\omega)}{2}$.

<u>Corollary:</u> The weak acoustic limit of the Boltzmann-Fermi-Dirac equation holds, for all $0<\tau<1$ and $k>2\tau$. (See later.)

<u>Remark:</u> - The acoustic limit holds for Dolbeault's solutions, with $b \in L^1 \cap L^\infty$.

- More interesting work to reach more general cross-sections.
- No restriction $k \leq 2$!!! Are we getting closer to understanding a compressible Euler limit?

The Relative Entropy near Absolute Zero

Equilibrium: $\quad M_\varepsilon(v) = \dfrac{\delta^{-1}}{1+\exp\left(\dfrac{v^2-R^2}{\varepsilon^\tau}\right)} \qquad (R>0, \quad \tau>0)$

Fluctuations: $\quad f_\varepsilon = M_\varepsilon + \varepsilon g_\varepsilon$

Relative entropy bound: $\quad (\gamma>0)$

$$\frac{1}{\varepsilon^{2-\gamma}} H_{FD}(f_\varepsilon|M_\varepsilon) = \frac{1}{\varepsilon^{2-\gamma}} \int_{\mathbb{R}^d\times\mathbb{R}^d} h(\delta f_\varepsilon, \delta M_\varepsilon)\,dx\,dv \leq C^{in} < \infty.$$

where $h(z,a) = z\log\dfrac{z}{a} + (1-z)\log\dfrac{1-z}{1-a}$, $(z,a)\in[0,1]\times(0,1)$.

Young-Fenchel inequalities, convex analysis:

The idea of using Young inequalities to extract a control of fluctuations from relative entropy bounds goes back to the work of Bardos-Golse-Levermore (1991).

Take E a real normed space, E^* its dual space, and a functional $f: D\subset E \to \mathbb{R}$. The Legendre transform (or Legendre-Fenchel transform) $f^*(y)$ of $f(z)$ is defined by

$$\boxed{f^*(y) = \sup_{z\in D}\left(\langle z,y\rangle_{E,E^*} - f(z)\right)}$$

on the dual domain

$$D^* = \left\{ y \in E^* \;\middle|\; \sup_{z \in D} \left(\langle z, y \rangle_{E,E^*} - f(z) \right) < \infty \right\}.$$

<u>Note:</u> D^* is convex, f^* is lower semi-continuous and convex (f^* is the supremum of affine functions.)

f^* is also called the convex conjugate of f.

<u>The Young-Fenchel inequality:</u>

$$\boxed{\begin{gathered} \langle z, y \rangle_{E,E^*} \leq f(z) + f^*(y). \\ \text{for all } z \in D, y \in D^*. \end{gathered}}$$

<u>Lemma:</u> For any $a \in (0,1)$, one has that

$$zy \leq \underbrace{\left[(a+z) \log\left(1 + \frac{z}{a}\right) + (1-a-z) \log\left(1 - \frac{z}{1-a}\right) \right]}_{= h(z+a,a)} + \left[\log\left(1 + a(e^y - 1)\right) - ay \right]$$

for all $z \in [-a, 1-a]$ and $y \in \mathbb{R}$.

More generally, for all $a \in (0,1)$, $z \in [-a, 1-a]$, $y \in \mathbb{R}$, and $\alpha > 0$, with $\alpha \varepsilon^{2-\gamma} \leq 1$, it holds that

$$|zy| \leq \frac{1}{\alpha \varepsilon^{2-\gamma}} h(a+z, a) + a(1-a) \alpha \varepsilon^{2-\gamma} \left(e^y + e^{-y} - 2 \right).$$

Proof: Fix $0<a<1$ and write $h_1(z) = h(a+z, a)$, $z \in [-a, 1-a]$.
Optimization of $z \mapsto zy - h_1(z)$, $y \in \mathbb{R}$, shows a global maximum at

$$z_* = \frac{a(1-a)(e^y - 1)}{1 + a(e^y - 1)}$$

$$\Rightarrow h_1^*(y) = z_* y - h_1(z_*) = \log\big(1 + a(e^y - 1)\big) - ay.$$

$$\Rightarrow zy \le h_1(z) + h_1^*(y). \quad \text{(Young-Fenchel inequality)}$$

Then, one can show that $h_1^{*\,\prime\prime}(y) \le a(1-a)(e^y + e^{-y})$

$$\Rightarrow h_1^*(y) \le a(1-a)\,(\underbrace{e^y + e^{-y} - 2}_{=\,w(y)})$$

Noticing that $w(\lambda y) \le \lambda^2 w(y)$, for all $\lambda \in [0,1]$, $y \in \mathbb{R}$,

$$\Rightarrow |zy\alpha| = \frac{1}{\varepsilon^{2-\gamma}}\,|zy\alpha\,\varepsilon^{2-\gamma}| \le \frac{1}{\varepsilon^{2-\gamma}}\left(h_1(z) + h_1^*\Big(\varepsilon^{2-\gamma}|y\alpha|\frac{z}{|z|}\Big)\right)$$

$$\underset{\alpha\varepsilon^{2-\gamma} \le 1.}{\le} \frac{1}{\varepsilon^{2-\gamma}}\,h_1(z) + a(1-a)\,\varepsilon^{2-\gamma}\,\alpha^2\,w(y).$$

$\blacksquare$

Proposition: If $\gamma + \tau \leq 2$, it holds that

$$g_\varepsilon = O\left(\varepsilon^{\frac{\tau-\gamma}{2}}\right)_{L^\infty(dt;\, L^1_{loc}(dx;\, L^1(dv)))}$$

and

$$g_\varepsilon |v^2 - R^2| = O\left(\varepsilon^{\frac{3\tau-\gamma}{2}}\right)_{L^\infty(dt;\, L^1_{loc}(dx;\, L^1(dv)))}$$

Proof: Setting $a = \delta M_\varepsilon$, $z = \delta \varepsilon g_\varepsilon$, $y = 1$, $\alpha = \varepsilon^{\frac{\gamma-\tau}{2}-1}$ in the Young-Fenchel inequality (need $\gamma+\tau \leq 2$) gives

$$\varepsilon^{\frac{\gamma-\tau}{2}} \delta |g_\varepsilon| \leq \underbrace{\frac{1}{\varepsilon^{2-\gamma}} h(\delta f_\varepsilon, \delta M_\varepsilon)}_{= O(1)_{L^\infty(dt;\, L^1(dx dv))}} + \underbrace{\delta M_\varepsilon (1 - \delta M_\varepsilon)}_{= O(\varepsilon^\tau)_{L^\infty(dt dx;\, L^1(dv))}} \varepsilon^{-\tau} w(1)$$

$$\Rightarrow \quad \varepsilon^{\frac{\gamma-\tau}{2}} g_\varepsilon = O(1)_{L^\infty(dt;\, L^1_{loc}(dx;\, L^1(dv)))}$$

Similarly, setting $y = \frac{|v^2 - R^2|}{2\varepsilon^\tau}$ instead of $y = 1$, gives

$$\varepsilon^{\frac{\gamma-3\tau}{2}} \delta |g_\varepsilon (v^2 - R^2)| \leq \frac{2}{\varepsilon^{2-\gamma}} h(\delta f_\varepsilon, \delta M_\varepsilon) + \underbrace{2\delta M_\varepsilon (1 - \delta M_\varepsilon) \varepsilon^{-\tau} w\left(\frac{v^2 - R^2}{2\varepsilon^\tau}\right)}_{= O(1)_{L^\infty(dt dx;\, L^1(dv))}}$$

$$\Longrightarrow \quad \varepsilon^{\frac{\gamma-3\tau}{2}} g_\varepsilon |v^2-R^2| = O(1)_{L^\infty(dt;\, L^1_{loc}(dx;\, L^1(dv)))}$$

$\blacksquare$

Refinements of the analysis of $h(z,a)$ show the following result.

<u>Proposition:</u> If $\gamma+\tau<2$, it holds that

$$\lim_{\lambda\to\infty} \sup_{\varepsilon,t} \int_{K\times\mathbb{R}^d} \varepsilon^{\frac{\gamma-3\tau}{2}} |g_\varepsilon(v^2-R^2)|\, \mathbb{1}_{\{|v^2-R^2|\geq\lambda\varepsilon^\tau\}}\, dx\,dv = 0$$

for any compact set $K\subset\mathbb{R}^d$.

Moreover, for any $\gamma>0$, $\tau>0$,

$$\sup_t \int_{\mathbb{R}^d\times\mathbb{R}^d} (g_\varepsilon)^2 \left(1+\frac{|v^2-R^2|}{\varepsilon^\tau}\right) dx\,dv \lesssim \varepsilon^{-\gamma}$$

<u>Remark:</u> From previous bounds, the entropy bound does not produce a meaningful control of fluctuations if $\gamma>\tau$. Whereas, a choice of parameters $\gamma<\tau$ leads to $g_\varepsilon\to 0$.

$$\Longrightarrow \quad 0<\gamma=\tau\leq 1$$

<u>Proposition</u>: Suppose that $0 < \gamma = \tau \leq 1$.

Up to extraction a subsequence, we have that

$$g_\varepsilon \xrightarrow{\ \ *\ \ } \mu \quad \text{in } \mathcal{M}_{loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d).$$

For any continuous function $\varphi(t,x,v)$ such that $\dfrac{\varphi(t,x,v)}{1+v^2} \in L^\infty(dt\,dx\,dv)$, compactly supported in (t,x), one has that the family

$$\int_0^\infty g_\varepsilon(t,x,r\sigma)\,\varphi(t,x,r\sigma)\,r^{d-1}\,dr$$

with $(t,x,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{S}^{d-1}$, is weakly relatively compact in $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{S}^{d-1})$.

Furthermore,

$$\int_{\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d} g_\varepsilon(t,x,v)\,\varphi(t,x,v)\,dt\,dx\,dv \longrightarrow \int_{\mathbb{R}^+\times\mathbb{R}^d\times\partial B(0,R)} g(t,x,w)\,\varphi(t,x,w)\,dt\,dx\,dw$$

that is, $\mu(t,x,v) = g\left(t,x,R\dfrac{v}{|v|}\right) dt \otimes dx \otimes \delta_{\partial B(0,R)}(v)$,

with $$\int_{\mathbb{R}^d\times\partial B(0,R)} g^2(t,x,w)\,dx\,dw \leq \frac{C^{in}}{R\delta^2}$$

(optimal constant).

Proof of L^2 bound:

Setting $z = \delta \varepsilon g_\varepsilon$, $a = \delta M_\varepsilon$, $y = \varepsilon^{1-\tau} \lambda \varphi(t, x, R\frac{v}{|v|})$, $\lambda > 0$,

in the Young-Fenchel inequality, and integrating in (t,x,v), gives

$$\delta\lambda \int_{[t_1,t_2]\times\mathbb{R}^d\times\mathbb{R}^d} g_\varepsilon \varphi \, dt\, dx\, dv \leq (t_2-t_1) C^{in}$$
$$+ \frac{1}{\varepsilon^{2-\tau}} \int \left[\log\left(1+\delta M_\varepsilon\left(e^{\varepsilon^{1-\tau}\lambda\varphi}-1\right)\right) - \delta M_\varepsilon \varepsilon^{1-\tau}\lambda\varphi \right] dt\, dx\, dv$$

$$\Rightarrow \delta\lambda \int_{[t_1,t_2]\times\mathbb{R}^d\times\partial B(0,R)} g(t,x,w)\varphi(t,x,w)\, dt\, dx\, dw \leq (t_2-t_1) C^{in}$$
$$+ (2R)^{-1} \int \underbrace{\left(\int_{-\infty}^{\infty} \left[\log\left(1+\frac{e^{\lambda\varphi}-1}{1+e^u}\right) - \frac{\lambda\varphi}{1+e^u} \right] du \right)}_{= \frac{(\lambda\varphi)^2}{2} \text{ (calculus miracle)}} dt\, dx\, dw$$

needs work!

Optimizing in $\lambda > 0$ gives

$$\delta \int g \varphi \, dt\, dx\, dw \leq \left(\frac{(t_2-t_1)C^{in}}{R} \right)^{1/2} \|\varphi\|_{L^2(dt\,dx\,dw)}$$

$\Rightarrow g \in L^2(dt\,dx\,dw)$. ■