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Hydrodynamic Regimes near Fermionic Condensates

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A Fermionic condensate is a superfluid state of matter formed by fermions at very low temperatures. In the Boltzmann–Fermi–Dirac equation, which governs the evolution of a Fermi gas, this superfluid phase is identified by an equilibrium state in which all particles occupy their lowest possible energy state, while obeying the Pauli exclusion principle. Our goal is to explore hydrodynamic regimes of the Boltzmann–Fermi–Dirac equation near a Fermionic condensate. In particular, we will discuss how the control of the relative entropy near absolute zero leads to singular velocity distributions which are concentrated on a sphere. The geometry of collisions in this quantized state will require special care and some new tools. We will also show how the analysis of the relative entropy near Fermionic condensates allows us to establish some hydrodynamic regimes of the Boltzmann–Fermi–Dirac equation, while opening doors to fresh new research perspectives.

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The Boltzmann Equation

Particle number density: $\quad f(t, x, v) \geq 0$

$t \in [0,\infty), \quad x \in \mathbb{R}^d, \quad v \in \mathbb{R}^d, \quad d \geq 2.$

Maxwellian distribution: $\quad f(t,x,v) = \dfrac{\rho(t,x)}{(2\pi\theta(t,x))^{d/2}} e^{-\frac{|v-u(t,x)|^2}{2\theta(t,x)}}$

Describes a statistical equilibrium in classical mechanics.

Note that: $\quad \rho = \int_{\mathbb{R}^d} f \, dv, \quad \rho u = \int_{\mathbb{R}^d} f \, v \, dv, \quad \rho\theta = \int_{\mathbb{R}^d} f \, \dfrac{|v-u|^2}{d} \, dv$

The Boltzmann equation:

$$\underbrace{(\partial_t + v \cdot \nabla_x)}_{\text{particles are transported by their own motion}} f(t,x,v) = \underbrace{Q_B(f)}_{\text{particles collide}}(t,x,v)$$

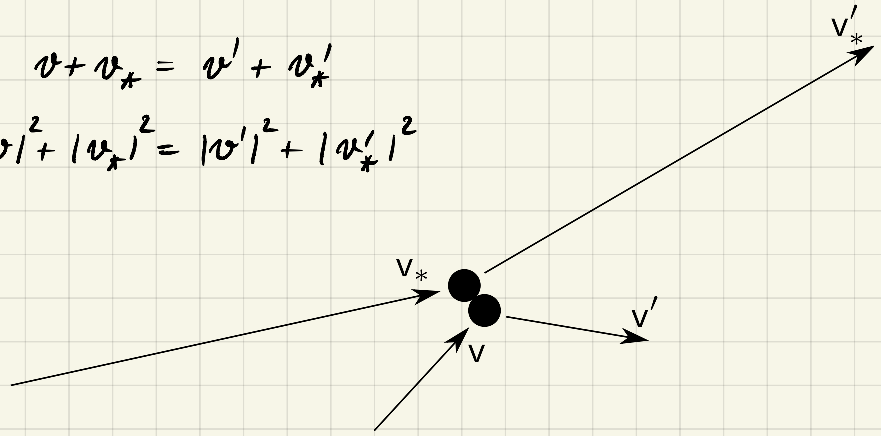
The Boltzmann collision operator:

$$Q_B(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \big(f(v') f(v'_*) - f(v) f(v_*) \big) \, b(v-v_*, \sigma) \, d\sigma \, dv_*$$

$$= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \big(f' f'_* - f f_* \big) \, b(v-v_*, \sigma) \, dv_* \, d\sigma$$

Conservation of momentum: $\quad v + v_* = v' + v'_*$

Conservation of energy: $\quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$

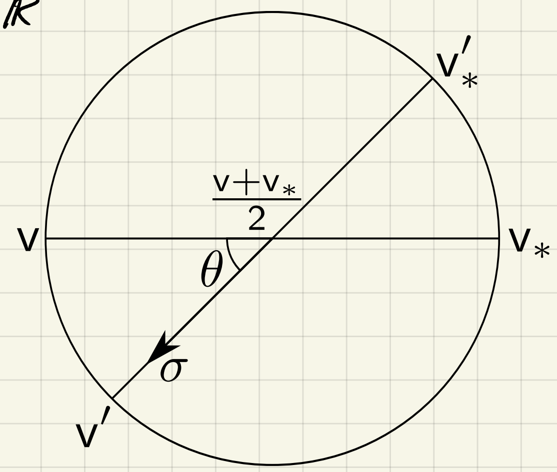


Given $v, v_* \in \mathbb{R}^d$, the solutions $v', v_*' \in \mathbb{R}^d$ can be parametrized by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$$

$$v_*' = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$$

where $\sigma \in \mathbb{S}^{d-1}$.



<u>Collision invariants:</u> The solutions $\phi(v)$ to the functional relation $\phi + \phi_* = \phi' + \phi_*'$ are linear combinations of 1, v, and $|v|^2$.

<u>Five hypotheses in the derivation of $Q(f,f)$:</u>

$$Q_B(f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (f' f_*' - f f_*)\, b \, dv_* \, d\sigma$$

- binary collisions (rarefied gas)
- localization in time and space of collisions
- elastic collisions
- micro-reversibility of collisions
- molecular chaos

<u>The collision kernel (cross-section):</u>

$$b(v-v_*, \sigma) = b(|v-v_*|, \cos\theta) \geq 0$$

It measures the "likelihood" of a collision with relative velocity $|v-v_*|$ and scattering angle θ.

The collision kernel will not be our focus.

Think that $b \equiv 1$ or $b \in L^1 \cap L^\infty$ whenever necessary

Conservation laws: $$\int_{\mathbb{R}^d} Q_B(f)(v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} dv = 0$$

Mass: $$\partial_t \int f\,dv + \nabla_x\cdot\int f\,v\,dv = 0 \Rightarrow \frac{d}{dt}\iint f\,dx\,dv = 0$$

Momentum: $$\partial_t \int f\,v\,dv + \nabla_x\cdot\int f\,v\otimes v\,dv = 0 \Rightarrow \frac{d}{dt}\iint f\,v\,dx\,dv = 0$$

Energy: $$\partial_t \int f\,|v|^2dv + \nabla_x\cdot\int f\,|v|^2 v\,dv = 0 \Rightarrow \frac{d}{dt}\iint f\,|v|^2\,dx\,dv \leq 0$$

Collisional symmetries: $$(v, v_*, \sigma) \longmapsto \left(v', v_*', \frac{v-v_*}{|v-v_*|}\right)$$

Induces a volume preserving pre-post-collisional change of variables.

$$\Rightarrow \int Q_B(f)(v)\,\psi(v)\,dv = \frac{1}{4}\int (f'f_*' - f f_*)(\psi+\psi_*-\psi'-\psi_*')\,b\,d\sigma\,dv_*\,d\sigma$$

Entropy dissipation:

$$\mathcal{D}_B(f) = -\int Q_B(f)(v)\log f(v)\,dv = \frac{1}{4}\int (f'f_*' - ff_*)\log\left(\frac{f'f_*'}{ff_*}\right) b\,dv\,dv_*\,d\sigma \geq 0.$$

In particular, $Q_B(f) = 0 \Rightarrow \mathcal{D}_B(f) = 0 \Rightarrow \log f$ is a collision invariant

$\Rightarrow f$ is a Maxwellian distribution, i.e., $f = \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v-u|^2}{2\theta}}$

H-theorem:

Global Maxwellian: $M(v) = \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v-u|^2}{2\theta}}$, with ρ, u, θ constant.

Relative entropy: $$H_B(f\,|\,M) = \int_{\mathbb{R}^d\times\mathbb{R}^d}\left(f\log\left(\frac{f}{M}\right) - f + M\right) dx\,dv \geq 0$$

$$H_B(f\,|\,M)(t) + \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_B(f)(s)\,dx\,ds \leq H_B(f_0\,|\,M).$$

Existence of renormalized solutions:

Theorem: (DiPerna-Lions, 1989, 1991)

Take $b(z,\sigma) \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$ and $f^{in} \in L^1_{loc}(dx\,;\, L^1((1+v^2)dv))$

such that $H(f_0 \mid M) < \infty$.

Then, there exists a renormalized solution $f(t,x,v)$ such that

$$\partial_t \int f\,dv + \nabla_x \cdot \int f v\,dv = 0$$

and $$H_B(f \mid M)(t) + \int_0^t \int \mathcal{D}_B(f)(s)\,dx\,ds \leq H_B(f^{in} \mid M).$$

Remark: - The control of the relative entropy implies that

$$f \in L^\infty(dt\,; L^1_{loc}(dx\,; L^1((1+|v|^2)dv)))$$

- No other local conservation laws are known to hold rigorously.
- Result holds for locally integrable cross-sections with quadratic growth in velocity.

The Boltzmann–Fermi–Dirac Equation

Fermions are quantum particles which satisfy the Pauli exclusion principle, i.e., particles which cannot occupy simultaneously the same quantum state.
A gas of Fermi particles at thermodynamic equilibrium can be shown to have energy states which are distributed according to Fermi-Dirac statistics.

<u>Fermi-Dirac distribution:</u>

$$f(t,x,v) = \frac{\delta^{-1}}{e^{\frac{|v-u|^2-\mu}{\theta}} + 1}$$

u is the bulk velocity, θ is the temperature, μ is the chemical potential, and $\delta > 0$ is related to Planck's constant.
Here, f is normalized so that $0 \le f \le \delta^{-1} \Leftrightarrow$ Pauli exclusion principle.

<u>The Boltzmann-Fermi-Dirac equation:</u>

$$(\partial_t + v.\nabla_x) f(t,x,v) = Q_{FD}(f)$$

$$Q_{FD}(f)(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \Big(f' f'_* (1-\delta f)(1-\delta f_*) - f f_* (1-\delta f')(1-\delta f'_*) \Big) b(v-v_*,\sigma)\, dv_* d\sigma$$

$\delta > 0$ is a multiple of the cube of Planck's constant.
New a priori estimate: $0 \le f \le \delta^{-1} \Rightarrow f \in L^\infty$

Classical limit: $Q_{FD}(f) \longrightarrow Q_B(f)$, as $\delta \to 0$, at least formally.

Same formal conservation laws as in the classical setting:

$$\int_{\mathbb{R}^d} Q_{FD}(f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

Entropy dissipation:

$$\mathcal{D}_{FD}(f) = -\delta \int Q_{FD}(f)(v) \log\left(\frac{\delta f}{1-\delta f}\right) dv$$

$$= \frac{\delta}{4} \int \Big(f' f_*' (1-\delta f)(1-\delta f_*) - f f_* (1-\delta f')(1-\delta f_*') \Big) \log\left(\frac{f' f_*' (1-\delta f)(1-\delta f_*)}{f f_* (1-\delta f')(1-\delta f_*')} \right) b \, dv \, dv_* \, d\sigma \geq 0.$$

In particular, $Q_{FD}(f) = 0 \Rightarrow \mathcal{D}_{FD}(f) = 0 \Rightarrow \log\left(\frac{\delta f}{1-\delta f}\right)$ is a collision invariant $\Rightarrow$ f is a Fermi-Dirac distribution, i.e., $\delta f = \dfrac{1}{e^{\frac{|v-u|^2-\mu}{\theta}} + 1}$

H-theorem:

Take a global Fermi-Dirac distribution M_{FD}.

Relative entropy: $H_{FD}(f \mid M_{FD}) = \displaystyle\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\delta f \log\left(\frac{\delta f}{\delta M}\right) + (1-\delta f) \log\left(\frac{1-\delta f}{1-\delta M}\right) \right) dx\, dv \geq 0$

$$H_{FD}(f \mid M_{FD}) + \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_{FD}(f)(s) \, dx \, ds \leq H_{FD}(f^{in} \mid M_{FD})$$

Existence and uniqueness of weak solutions:

Theorem: (Dolbeault, 1994)

Take $b(z,\sigma) \in L^1(\mathbb{R}^d \times \mathbb{S}^{d-1})$
and $f^{in} \in L^\infty(dx\,dv) \cap L^1_{loc}(dx; L^1((1+|v|^2)dv))$
such that $H_{FD}(f^{in} \mid M_{FD}) < \infty$.
Then, there exists a unique weak solution $f \in L^\infty(dt\,dx\,dv)$, with $0 \leq f \leq \delta^{-1}$; and such that

$$\partial_t \int f \, dv + \nabla_x \cdot \int f \, v \, dv = 0,$$

$$\partial_t \int f \, v \, dv + \nabla_x \cdot \int f \, v \otimes v \, dv = 0,$$

and $$H_{FD}(f \mid M_{FD}) + \int_0^t \int D_{FD}(f)(s) \, dx \, dv \leq H_{FD}(f^{in} \mid M_{FD}).$$

Remark: - The control of the relative entropy implies that

$$f \in L^\infty(dt\, ; L^1_{loc}(dx\, ; L^1((1+|v|^2)\, dv))).$$

- Lions showed (1994) how to extend Dolbeault's result to more general cross-sections (with quadratic growth in velocity). But we lose uniqueness and conservation laws in this setting.

Classical Hydrodynamic Regimes

$$\left(\partial_t + v\cdot\nabla_x\right) f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon)$$ Boltzmann or Fermi–Dirac operator.

Knudsen number = $\dfrac{\text{mean-free path}}{\text{length scale}}$

Other interpretation: hyperbolic scaling $f_\varepsilon(t,x,v) = f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$

$\varepsilon \to 0$ is a continuum limit, we expect a macroscopic description of the gas. Formally:

if $f_\varepsilon \to f$ and $Q(f_\varepsilon) \longrightarrow Q(f)$, as $\varepsilon \to 0$
then $Q(f) = 0$
$\Rightarrow$ f is a Maxwellian or a Fermi–Dirac distribution.

Setting $\rho = \int f\,dv$, $\rho u = \int f v\,dv$, $\rho\theta = \int f \frac{|v-u|^2}{d}\,dv$,

the local conservation laws yield the compressible Euler system:

$$\begin{cases} \partial_t \rho + \nabla_x\cdot(\rho u) = 0 & \text{(mass)} \\ \partial_t \rho u + \nabla_x\cdot\left(\rho u\otimes u + \rho\theta\, \mathrm{Id}\right) = 0 & \text{(momentum)} \\ \partial_t\left(\frac{1}{2}\rho u^2 + \frac{d}{2}\rho\theta\right) + \nabla_x\cdot\left(\frac{1}{2}\rho u\left(|u|^2 + (d+2)\theta\right)\right) = 0 & \text{(energy)} \end{cases}$$

General rigorous results justifying the compressible Euler limit remain challenging and, in many cases, out of reach.

Study the linearization of the compressible Euler system

$\Rightarrow$ acoustic waves.

Take fluctuations of order ε: $\rho = 1+\varepsilon\tilde{\rho}$, $u = \varepsilon\tilde{u}$, $\theta = 1+\varepsilon\tilde{\theta}$.

In the limit $\varepsilon \to 0$, we obtain the acoustic waves system:

$$\begin{cases} \partial_t \tilde{\rho} + \nabla_x \cdot \tilde{u} = 0 \\ \partial_t \tilde{u} + \nabla_x (\tilde{\rho} + \tilde{\theta}) = 0 \\ \frac{d}{2}\partial_t (\tilde{\rho}+\tilde{\theta}) + \frac{d+2}{2} \nabla_x \cdot \tilde{u} = 0 \end{cases}$$

<u>The acoustic limit of the Boltzmann equation:</u>

Normalized Maxwellian: $M(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{v^2}{2}}$

$$(\partial_t + v.\nabla_x) f_\varepsilon = \frac{1}{\varepsilon^k} Q_B(f_\varepsilon), \quad f_\varepsilon = M(1+\varepsilon g_\varepsilon)$$

$$\Rightarrow (\partial_t + v.\nabla_x) g_\varepsilon = \frac{-1}{\varepsilon^k} \mathcal{L}(g_\varepsilon) + \varepsilon^{1-k} \tilde{Q}(g_\varepsilon)$$

Suppose formally that $g_\varepsilon \longrightarrow g$ and $\mathcal{L}(g_\varepsilon) \longrightarrow \mathcal{L}(g)$

$$\Rightarrow \mathcal{L}(g) = \int_{\mathbb{R}^d \times S^{d-1}} (g + g_* - g' - g'_*)\, b(v-v_*,\sigma) M_* \, dv_* \, d\sigma = 0$$

One can show that the kernel of $\mathcal{L}$ is the linear span of collision invariants $\{1, v_1, v_2, \dots, v_d, |v|^2\}$.

$$\Rightarrow \boxed{g(t,x,v) = \rho(t,x) + v\cdot u(t,x) + \left(\frac{|v|^2-d}{2}\right)\theta(t,x)}$$

Taking the formal limit in conservations laws shows that (ρ, u, θ) solves the acoustic waves system.

<u>Theorem:</u> (Weak Acoustic Limit Theorem)

(Bardos, Golse, Levermore, Jiang, Masmoudi, 1991–2010)

Suppose that $k \leq 2$ and f_ε is a renormalized solution, for each $\varepsilon > 0$, such that

$$\boxed{\frac{1}{\varepsilon^2} H_B(f_\varepsilon | M)(t) + \frac{1}{\varepsilon^{2+k}} \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_B(f)(s)\, dx\, ds \leq \frac{1}{\varepsilon^2} H_B(f_\varepsilon^{in} | M) \leq C^{in} < \infty}$$

Then, as $\varepsilon \to 0$, $g_\varepsilon \to \rho + v\cdot u + \left(\frac{|v|^2 - d}{2}\right)\theta$ in a suitable weak sense (at least in the sense of distributions), where

$$(\rho, u, \theta) \in C([0,\infty); L^2(dx))$$

solves the acoustic waves system.

<u>Remarks:</u> - The restriction $k \leq 2$ is necessary to control the conservation defects. Related to the lack of control of large velocities. The range $k > 2$ remains an important open problem.

- A similar theorem is expected to hold for the acoustic limit of the Boltzmann–Fermi–Dirac equation near a global Fermi–Dirac distribution: $f_\varepsilon = M_{FD} + \varepsilon \delta M_{FD}(1 - \delta M_{FD}) g_\varepsilon$ (See Zakrevskiy, PhD thesis, for formal derivations.)

- In particular, we do not observe quantum effects in these hydrodynamic regimes.

Hydrodynamic Regimes near Absolute Zero Temperature

Idea: Explore hydrodynamic regimes of the Boltzmann-Fermi-Dirac equation to uncover interesting quantum effects.

Fermionic condensate

There is another interesting global equilibrium state of the Boltzmann-Fermi-Dirac equation.

In the zero-temperature limit of a Fermi-Dirac distribution, we obtain a Fermionic condensate:

$$M_{FD}(v) = \frac{\delta^{-1}}{1+\exp\left(\frac{|v-u|^2-R^2}{\varepsilon^2}\right)} \xrightarrow[(R>0)]{\varepsilon\to 0} F(v) = \delta^{-1}\mathbb{1}_{B(u,R)}(v)$$

Fermionic condensates are equilibrium states where particles are at their minimum energy state while satisfying Pauli exclusion principle. They are characteristic functions of balls.

Note that $f f_*(1-\delta f')(1-\delta f_*') - f' f_*'(1-\delta f)(1-\delta f_*) = 0$ if $\delta f = \mathbb{1}_{B(u,R)}(v)$.